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Abstract. A new class of electromagnetic fields within the realm of the axially symmetric gravitational fields is obtained. It is shown that this class of fields is obtained as a solution of the Painlevé differential equation.

Various classes of solution of the Einstein–Maxwell field equations in an axially symmetric static space–time have been discussed in the literature (Misra and Radhakrishna 1962, Harrison 1965, 1968, Misra 1970, Gautreau and Hoffman 1970, Misra and Pandey 1971, Misra *et al* 1973). In this paper we outline a method by which one can generate a new class of electromagnetic fields which is different from all previously known solutions.

Consider the following set of field equations:

$$R_{ij}-\tfrac{1}{2}g_{ij}R = G_{ij} = -8\pi E_{ij}, \tag{1}$$

$$F^{ij}{}_{;j} = 0, \tag{2}$$

$$F_{[ij;k]} = 0 \tag{3}$$

where

$$E_{ij} = \frac{1}{4\pi}(g^{kl}F_{ik}F_{jl}-\tfrac{1}{4}g_{ij}F_{kl}F^{kl}). \tag{4}$$

The symbols used here have their usual meaning. Let us consider the axisymmetric static metric in the form:

$$ds^2 = e^{2u}\,dt^2 - e^{2k-2u}(d\rho^2+dz^2)-\rho^2\,e^{-2u}\,d\phi^2 \tag{5}$$

where u and k are functions of ρ and z only.

Now, the field equations (1) can be obtained if the components of E_{ij} are known. The components of the Maxwell tensor can be obtained in this case with the help of only two components of the four-potential. If one introduces the potentials A and B in the following manner (Harrison 1968, Misra 1970):

$$\left.\begin{aligned} F^{\alpha\beta} &= (-g)^{-1/2}\epsilon^{\alpha\beta\gamma}A_{,\gamma} \\ F_{0\alpha} &= B_{,\alpha} \end{aligned}\right\} \qquad \alpha, \beta = 1, 2, 3 \tag{6}$$

where $A = A(\rho, z)$ and $B = B(\rho, z)$, one obtains the stress tensor E_{ij} in a symmetrical form with respect to A and B. Further, there are only two nontrivial Maxwell equations amongst (2) and (3) which are identical with regard to A and B. This situation leads one to introduce a single potential C (ie duality-rotation) such that:

$$A = C\sin\lambda \qquad B = C\cos\lambda \tag{7}$$

where λ is a constant. These considerations lead to the following field and Maxwell equations:

$$u_{,11}+u_{,22}+\frac{u_{,1}}{\rho} = -e^{-2u}(C_{,1}{}^2+C_{,2}{}^2), \tag{8}$$

$$C_{,11}+C_{,22}+\frac{C_{,1}}{\rho}-2(C_{,1}u_{,1}+C_{,2}u_{,2}) = 0, \tag{9}$$

$$\frac{k_{,1}}{\rho} = (u_{,1}{}^2-u_{,2}{}^2)+e^{-2u}(C_{,1}{}^2-C_{,2}{}^2), \tag{10}$$

$$\frac{k_{,2}}{\rho} = 2u_{,1}u_{,2}+2e^{-2u}C_{,1}C_{,2}. \tag{11}$$

Let us introduce a new function a in the following manner:

$$C_{,1} = \frac{a_{,2}}{\rho}e^{2u}, \qquad C_{,2} = -\frac{a_{,1}}{\rho}e^{2u}. \tag{12}$$

In view of equations (12), equation (9) is identically satisfied and its integrability conditions and the remaining equations (8)–(11) yield:

$$\nabla^2 u+\frac{e^{2u}}{\rho^2}(a_{,1}{}^2+a_{,2}{}^2) = 0, \tag{13}$$

$$\nabla^2 a-\frac{2a_{,1}}{\rho}+2(a_{,1}u_{,1}+a_{,2}u_{,2}) = 0, \tag{14}$$

$$\frac{k_{,1}}{\rho} = (u_{,1}{}^2-u_{,2}{}^2)+\frac{e^{2u}}{\rho^2}(a_{,2}{}^2-a_{,1}{}^2), \tag{15}$$

$$\frac{k_{,2}}{\rho} \equiv 2u_{,1}u_{,2}-\frac{2a_{,1}a_{,2}}{\rho^2}e^{2u}, \tag{16}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}$$

is the two-dimensional laplacian operator. These equations may be expressed in an alternative form by the introduction of a scalar field $P(\rho, z)$ and a vector field $\boldsymbol{Q}(\rho, z)$ defined by

$$P(\rho, z) = \ln\rho-u, \tag{17}$$

$$\boldsymbol{Q}(\rho, z) = e^{-P}\,\mathrm{grad}\,a, \tag{18}$$

where grad is the usual vector operator defined with respect to the three-dimensional flat-space metric.

Equations (13) and (14) now take the form

$$\nabla^2 P = \boldsymbol{Q} \cdot \boldsymbol{Q}, \tag{19}$$

$$\operatorname{div} \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{L}, \tag{20}$$

$$\operatorname{curl} \boldsymbol{Q} = \boldsymbol{Q} \times \boldsymbol{L}, \tag{21}$$

where (21) arises from the integrability condition $a_{,12} = a_{,21}$ and $\boldsymbol{L} = \operatorname{grad} P$.

Equations (15) and (16) may be written in the vector form

$$\operatorname{curl} \boldsymbol{K} = 2\left\{-\left(\frac{1}{\rho} - P_{,1}\right)P_{,2} - Q_1 Q_2\right\}\boldsymbol{\rho} + \left\{P_{,2}{}^2 - \left(\frac{1}{\rho} - P_{,1}\right)^2 + Q_1^2 - Q_2^2\right\}\boldsymbol{Z} \tag{22}$$

where $\boldsymbol{\rho}$, $\boldsymbol{\phi}$ and $\boldsymbol{Z}$ are unit vectors along the corresponding coordinate curves and

$$\boldsymbol{Q} = \boldsymbol{\rho} Q_1 + \boldsymbol{Z} Q_2, \qquad \boldsymbol{K} = \frac{1}{\rho} k \boldsymbol{\phi}.$$

One obtains from (19)–(21), if $Q \neq 0$,

$$\operatorname{grad} P = \frac{1}{Q^2}\{(\operatorname{div} \boldsymbol{Q})Q - \boldsymbol{Q} \times \operatorname{curl} \boldsymbol{Q}\}. \tag{23}$$

Various classes of known electromagnetic fields are obtained by specifying particular relations between $\boldsymbol{L}$ and $\boldsymbol{Q}$, for example, if $\boldsymbol{L} \cdot \boldsymbol{Q} = (1/\rho)Q_1$, the Weyl class of electromagnetic fields is obtained (Misra and Radhakrishna 1962, Misra *et al* 1973).

A new class of electromagnetic fields is obtained by choosing $\boldsymbol{Q} = \boldsymbol{\rho} f(\rho) + \boldsymbol{Z} h(\rho)$, where f and h are functions of ρ only. From equation (23) one finds

$$(f^2 + h^2)\boldsymbol{L} = \boldsymbol{\rho}\left(ff_{,1} + \frac{f^2}{\rho} - hh_{,1}\right) + \boldsymbol{Z}\left((fh)_{,1} + \frac{fh}{\rho}\right).$$

With the help of the above expressions for $\boldsymbol{Q}$ and $\boldsymbol{L}$ it only remains to satisfy the equations $\operatorname{curl} \boldsymbol{L} = 0$ and $\operatorname{div} \boldsymbol{L} = Q^2$, which become

$$(fh)_{,1} + \frac{fh}{\rho} = D(f^2 + h^2),$$

$$\rho(f^2 + h^2) = \frac{\partial}{\partial \rho}\left(\frac{\rho ff_{,1} + f^2 - \rho hh_{,1}}{f^2 + h^2}\right)$$

where D is a constant. By eliminating $f^2 + h^2$ and integrating one gets two equations:

$$\begin{aligned} f_{,1} &= fh\left(\frac{D}{f} + \frac{f}{D}\right) - \frac{D + l}{D\rho} f, \\ h_{,1} &= Df - \frac{fh^2}{D} + \frac{hl}{D\rho}, \end{aligned} \tag{24}$$

where l is a second arbitrary constant. Now, it is possible to eliminate either of the independent variables from (24). Elimination of h from (24) yields the second order equation

$$(D^2 + f^2)(f_{,1} + \rho f_{,11}) - \rho ff_{,1}^2 = (D^2 + f^2)^2 \rho f + \frac{(D + l)^2 f}{\rho}. \tag{25}$$

Further, on making the substitutions $R = f^2/(D^2+f^2)$ and $S = \rho^2$ equation (25) reduces to

$$\frac{d^2R}{dS^2} = \left(\frac{dR}{dS}\right)^2\left(\frac{1}{2R}+\frac{1}{R-1}\right) - \frac{1}{S}\frac{dR}{dS} + \frac{D^2R}{2S} + \frac{(D+l)^2R(1-R)^2}{2D^2S^2} \tag{26}$$

which is a standard differential equation with fixed critical points defining a Painlevé transcendent (Ince 1927).

Thus, from equations (26) and (24) we can calculate the values of f and h. Once f and h are known, the solution is easily obtained.

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